

# ON THE GONALITY SEQUENCE OF SMOOTH CURVES: NORMALIZATIONS OF SINGULAR CURVES IN A QUADRIC SURFACE

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**ABSTRACT.** Let  $C$  be a smooth curve of genus  $g$ . For each positive integer  $r$  the  $r$ -gonality  $d_r(C)$  of  $C$  is the minimal integer  $t$  such that there is  $L \in \text{Pic}^t(C)$  with  $h^0(C, L) = r + 1$ . In this paper for all  $g \geq 40805$  we construct several examples of smooth curves  $C$  of genus  $g$  with  $d_3(C)/3 < d_4(C)/4$ , i.e. for which a slope inequality fails.

## 1. INTRODUCTION

Let  $C$  be a smooth and connected projective curve of genus  $g \geq 3$ . For each integer  $r \geq 1$  the  $r$ -gonality  $d_r(C)$  of  $C$  is the minimal integer  $d$  such that there is a degree  $d$  line bundle  $L$  on  $C$  with  $h^0(C, L) \geq r + 1$  ([3]). The sequence  $\{d_r(C)\}_{r \geq 1}$  is called the *gonality sequence* of  $C$ . This sequence is important to understand the Brill-Noether theory of vector bundles on  $C$  ([4], [5], [6]). See [3], §3, for general properties of this sequence for an arbitrary curve  $C$ . For most curves we have

$$(1) \quad \frac{d_r(C)}{r} \geq \frac{d_{r+1}(C)}{r+1}$$

for all  $r \geq 2$  ([3], Proposition 4.1). In [3] H. Lange and G. Martens introduced the following notion. The curve  $C$  is said to satisfy the *slope inequality* if (1) is satisfied for all  $r \geq 2$ . Since  $d_2(C) \leq 2d_1(C)$  for all  $C$ , the slope inequality is always satisfied for  $r = 1$ . Hence  $C$  does not satisfy the slope inequality if and only if there is at least one integer  $r \geq 2$  for which (1) fails. Many different examples of such curves are constructed in [3]. In this paper we look at the case  $r = 3$  of (1) and prove the following result.

**Theorem 1.** *Fix an integer  $g \geq 40805$ . Then there exists a smooth curve  $C$  of genus  $g$  such that  $d_3(C)/3 < d_4(C)/4$ .*

The curves  $C$  used to prove Theorem 1 are the normalization of nodal curves  $Y$  contained in a smooth quadric surface  $Q \subset \mathbb{P}^3$ . These families of examples are an extension of [3], Example 4.12. We prove that for the normalization of many of them the rational number  $d_4(C)/4 - d_3(C)/3$  is rather large (Propositions 1, 2 and Corollary 1). As an obvious consequence we get the following statement.

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**Theorem 2.** *There is a sequence  $\{C_g\}_{g \geq 3}$  of smooth curves such that  $C_g$  has genus  $g$ ,*

$$\lim_{g \rightarrow \infty} \frac{d_4(C_g)}{d_3(C_g)} = 3/2 \text{ and } \lim_{g \rightarrow \infty} \frac{d_4(C_g)/4 - d_3(C_g)/3}{\sqrt{g}} = 1/12.$$

To prove Theorems 1 and 2 we need to study the cohomology of certain finite subsets  $S \cup B$  of the smooth quadric surface  $Q$ . These preliminary lemmas are proved in section 2. In section 3 we use these lemmas in the following way. Fix an integral nodal curve  $Y \in |\mathcal{O}_Q(a, a + m)|$  and set  $S := \text{Sing}(Y)$ . Let  $C$  be the normalization of  $Y$ . Fix any  $L \in \text{Pic}^z(C)$  evincing  $d_4(C)$ . To a general divisor  $A \in |L|$  we associate a set  $B \subset Q \setminus S$  such that  $\sharp(B) = z$  and  $h^1(Q, \mathcal{I}_{S \cup B}(a - 2, a + m - 2)) > 0$ . The lemmas proved in section 2 show that  $z \geq 3a - 15$  for a general  $S$  and  $m$  not too large, while obviously  $d_3(C) \leq 2a$ . Taking only smooth curves inside  $Q$  we only get a sequence of genera, enough to prove the weaker form of Theorem 2 with “lim sup” instead of “lim” (as implicit in [3], Example 4.12).

For all integers  $r \geq 2$  and  $g \geq 2$  let  $\alpha(r, g)$  be the supremum of all rational numbers  $d_{r+1}(C)/d_r(C)$  with  $C$  a smooth curve of genus  $g$ .

**Question 1.** Compute  $\alpha'(r) := \liminf_{g \rightarrow \infty} \alpha(r, g)$  and  $\alpha''(r) := \limsup_{g \rightarrow \infty} \alpha(r, g)$ . Is  $\alpha''(3) = 3/2$ ?

We work over an algebraically closed base field with characteristic zero.

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## 2. PRELIMINARIES

Let  $Q \subset \mathbb{P}^3$  be a smooth quadric surface. For any coherent sheaf  $\mathcal{F}$  on  $Q$  and any  $i \in \mathbb{N}$  set  $H^i(\mathcal{F}) := H^i(Q, \mathcal{F})$  and  $h^i(\mathcal{F}) := \dim(H^i(\mathcal{F}))$ . For all  $(a, b) \in \mathbb{Z}^2$  let  $\mathcal{O}_Q(a, b)$  denote the line bundle on  $Q$  with bidegree  $(a, b)$ . We have  $h^0(\mathcal{O}_Q(a, b)) = (a + 1)(b + 1)$  and  $h^1(\mathcal{O}_Q(a, b)) = 0$  if  $(a, b) \in \mathbb{N}^2$ , while  $h^0(\mathcal{O}_Q(a, b)) = 0$  if either  $a < 0$  or  $b < 0$ . If  $(a, b) \in \mathbb{N}^2$  and  $T \in |\mathcal{O}_Q(a, b)|$ , then we say that  $T$  has type  $(a, b)$ . The lines contained in  $Q$  are the curves  $D \subset Q$  with either type  $(1, 0)$  or type  $(0, 1)$ .

**Remark 1.** Fix a integer  $x > 0$  and a general  $S \subset Q$  such that  $\sharp(S) = x$ . Since  $h^0(\mathcal{O}_Q(1, 1)) = 4$ ,  $h^0(\mathcal{O}_Q(2, 1)) = h^0(\mathcal{O}_Q(1, 2)) = 6$  and  $S$  is general, we have  $\sharp(S \cap T_1) \leq 3$  for every  $T_1 \in |\mathcal{O}_Q(1, 1)|$ ,  $\sharp(S \cap T_2) \leq 5$  for every  $T_2 \in |\mathcal{O}_Q(2, 1)|$  and  $\sharp(S \cap T_3) \leq 5$  for every  $T_3 \in |\mathcal{O}_Q(1, 2)|$ .

**Lemma 1.** *Fix integers  $u > 0$  and  $v > 0$ . Fix a reduced  $D \in |\mathcal{O}_Q(2, 1)|$  and a set  $S \subset D$  such that  $\sharp(S) \leq 2v + u + 1$ ,  $\sharp(S \cap T) \leq 1$  for every line  $T \subset D$  (if any) and  $\sharp(S \cap T) \leq u + v + 1$  for every component  $T$  of type  $(1, 1)$  of  $D$  (if any). Then  $h^1(D, \mathcal{I}_{S, D}(u, v)) = 0$ .*

*Proof.* First assume that  $D$  is irreducible. Since  $D \cong \mathbb{P}^1$  and  $\deg(\mathcal{O}_D(u, v)) = 2v + u \geq \sharp(S) - 1$ , we have  $h^1(D, \mathcal{I}_{S, D}(u, v)) = 0$ . Now assume that  $D$  has an irreducible component  $A$  of type  $(1, 1)$  and write  $D = A \cup T$  with  $T$  a line of type  $(1, 0)$ . Since  $\sharp(A \cap S) \leq u + v + 1$ ,  $A \cong \mathbb{P}^1$  and  $\deg(\mathcal{O}_A(u, v)) = u + v$ , we have  $h^1(A, \mathcal{I}_{S \cap A, A}(u, v)) = 0$ . Since  $\sharp(T \cap S) \leq 1$  and  $S \subset D$ , we have  $\sharp(S \setminus S \cap A) \leq 1$ . Since  $\deg(T \cap A) = 1$ , we also have  $h^1(T, \mathcal{I}_{(S \setminus S \cap A) \cup (A \cap T), T}(u, v)) = 0$ . Hence a Mayer-Vietoris exact sequence gives  $h^1(D, \mathcal{I}_{S, D}(u, v)) = 0$ . Now assume that  $D$

is the union of 3 lines  $T_1, T_2, T_3$  with  $T_2$  of type  $(0, 1)$ . Since  $\sharp(T_i \cap S) \leq 1$  for all  $i$ , we have  $h^1(T_1, \mathcal{I}_{S \cap T_1}(u, v)) = 0$ ,  $h^1(T_2, \mathcal{I}_{S \cap T_2 \setminus S \cap T_1 \cap T_2, T_2}(u-1, v)) = 0$  and  $h^1(T_3, \mathcal{I}_{S \setminus (T_1 \cup T_2) \cap S, T_3}(u-1, v-1)) = 0$ . We use two Mayer-Vietoris exact sequences and get first  $h^1(T_1 \cup T_2, \mathcal{I}_{S \cap (T_1 \cup T_2), T_1 \cup T_2}(u, v)) = 0$  and then  $h^1(D, \mathcal{I}_{S, D}(u, v)) = 0$ .  $\square$

**Lemma 2.** *Fix integers  $v \geq u \geq 9$  and set  $\alpha := \lfloor u/3 \rfloor$ . Fix a finite set  $E \subset Q$  such that  $\sharp(E) \leq v - u + 10\alpha$ , no 2 of the points of  $E$  are contained in a line of  $Q$ , at most  $2u + 1$  of the points of  $E$  are contained in a curve of type  $(1, 1)$ , at most  $3u + 1$  of the points of  $E$  are contained in a curve of type  $(2, 1)$  and at most  $3u - 4$  of the points of  $E$  are contained in a curve of type  $(1, 2)$ . Then  $h^1(\mathcal{I}_E(u, v)) = 0$ .*

*Proof.* Notice that  $\alpha \geq 3$ . Set  $\beta := u - 3\alpha$

(i) In this step we assume  $v = u$ . Set  $E_0 := E$ . Take any  $A_1 \in |\mathcal{O}_Q(2, 1)|$  such that  $a_1 := \sharp(E_0 \cap A_1)$  is maximal. Set  $F_1 := A_1 \cap E_0$  and  $E_{1,0} := E_0 \setminus F_1$ . Let  $D_1 \in |\mathcal{O}_Q(1, 2)|$  be a curve such that  $b_1 := \sharp(E_{1,0} \cap D_1)$  is maximal. Set  $G_1 := E_{1,0} \cap D_1$  and  $E_1 := E_{1,0} \setminus G_1$ . For each  $i \in \{2, \dots, \alpha\}$  we define recursively the integers  $a_i$  and  $b_i$ , the curves  $A_i \in |\mathcal{O}_Q(2, 1)|$ ,  $D_i \in |\mathcal{O}_Q(1, 2)|$  and the sets  $F_i$ ,  $E_{i,0}$ ,  $G_i$ ,  $E_i$  in the following way. Take  $A_i \in |\mathcal{O}_Q(2, 1)|$  such that  $a_i := \sharp(A_i \cap E_{i-1})$  is maximal. Set  $F_i := E_{i-1} \cap A_i$  and  $E_{i,0} := E_{i-1} \setminus F_i$ . Take  $D_i \in |\mathcal{O}_Q(1, 2)|$  such that  $b_i := \sharp(D_i \cap E_{i,0})$  is maximal and set  $G_i := D_i \cap E_{i,0}$  and  $E_i := E_{i,0} \setminus G_i$ . For each  $i \in \{1, \dots, \alpha\}$  we have the exact sequences

$$(2) \quad \begin{aligned} 0 &\rightarrow \mathcal{I}_{E_{i,0}}(u - 3i + 1, u - 3i + 2) \rightarrow \mathcal{I}_{E_{i-1}}(u - 3i + 3, u - 3i + 3) \\ &\rightarrow \mathcal{I}_{F_i, A_i}(u - 3i + 3, u - 3i + 3) \rightarrow 0 \end{aligned}$$

$$(3) \quad \begin{aligned} 0 &\rightarrow \mathcal{I}_{E_i}(u - 3i, u - 3i) \rightarrow \mathcal{I}_{E_{i,0}}(u - 3i + 1, u - 3i + 2) \\ &\rightarrow \mathcal{I}_{G_i, D_i}(u - 3i + 1, u - 3i + 2) \rightarrow 0 \end{aligned}$$

Notice that the sequences  $\{a_i\}_{1 \leq i \leq \alpha}$  and  $\{b_i\}_{1 \leq i \leq \alpha}$  are non-increasing. If  $a_i \leq 4$ , then  $E_{i,0} = \emptyset$ , because  $h^0(Q, \mathcal{O}_Q(2, 1)) = 6$ . If  $b_i \leq 4$ , then  $E_i = \emptyset$ , because  $h^0(Q, \mathcal{O}_Q(1, 2)) = 6$ . Since  $\sharp(E) \leq 10\alpha$ , we get  $E_\alpha = \emptyset$ .

*Claim 1:* For every  $i \in \{1, \dots, \alpha\}$  we have  $a_i \leq 3u - 9i + 10$ .

*Proof of Claim 1:* Assume  $a_i \geq 3u - 9i + 11$ . Since at most  $3u + 1$  of the points of  $E$  are contained in a curve of type  $(2, 1)$ , we have  $i \geq 2$ . Since the sequences  $\{a_n\}$ ,  $\{b_n\}$  are non-increasing and  $b_j \geq 5$  if  $a_{j+1} > 0$ , we get  $\sharp(E) \geq i(3u - 9i + 11) + 5(i - 1) = i(3u + 16 - 9i) - 5$ . If  $i = 2$ , then we get  $\sharp(E) \geq 6u - 9 > 10\alpha$ , a contradiction. For any  $t \in \mathbb{R}$  set  $\phi(t) = t(3u + 16 - 9t) - 5$ . The function  $\phi$  is increasing in the interval  $[0, (3u + 16)/18]$  and decreasing if  $t \geq (3u + 16)/18$ . Since  $\sharp(E) < \phi(2)$  and  $\phi(u/3) = 16u/3 - 5 > \sharp(E)$ , we get a contradiction.

*Claim 2:* For each  $i \in \{1, \dots, \alpha\}$  we have  $h^1(A_i, \mathcal{I}_{F_i, A_i}(u - 3i + 3, u - 3i + 3)) = 0$ .

*Proof of Claim 2:* By Claim 1 we have  $\sharp(F_i) \leq 3u - 9i + 10$ . If  $A_i$  is irreducible, then Claim 2 is true (e.g. by Lemma 1). If  $A_i$  is the union of 3 lines, then  $a_i \leq 3$  and Claim 2 is true (Lemma 1). Now assume  $A_i = T \cup D$  with  $T$  a smooth conic and  $D$  of type  $(1, 0)$ . By Lemma 1 Claim 2 is true if  $\sharp(F_i \cap T) \leq 2u - 6i + 7$ . Assume  $\sharp(F_i \cap T) \geq 2u - 6i + 8$ . Our assumptions on  $E$  imply  $i \geq 2$ . We have  $a_i \geq \sharp(A_i \cap T) \geq 2u - 6i + 8$ . Since  $b_j \geq 5$  if  $E_{j+1} \neq \emptyset$ , we get  $\sharp(E) \geq i(2u - 6i + 8) + 5(i - 1) = i(2u + 13 - 6i) - 5$ . If  $i = 2$ , then  $\sharp(E) \geq 4u - 3 > 10\alpha$ , a contradiction. For every  $t \in \mathbb{R}$  set  $\psi(t) := t(2u + 13 - 6t) - 5$ . Since the function

$\psi$  is increasing in the interval  $[0, (2u+13)/12]$  and decreasing for  $t > (2u+13)/12$ ,  $\sharp(E) < \psi(2)$  and  $\psi(\alpha) \geq 13\alpha - 5 > 10\alpha$ , we get a contradiction.

*Claim 3:* For each  $i \in \{1, \dots, \alpha\}$  we have  $b_i \leq 3u - 9i + 5$ .

*Proof of Claim 3:* Assume  $b_i \geq 3u - 9i + 6$ . Since  $G_i \subseteq E$ , our assumptions on  $E$  imply  $i \geq 2$ . Since  $b_i > 0$ , we have  $a_j \geq 5$  for all  $j \leq i$ . Hence  $\sharp(E) \geq 5i + i(3u - 9i + 6) = i(3u + 11 - 9i)$ . Set  $\tau(t) = t(3u + 11 - 9t)$ . The function  $\tau(t)$  is increasing in the interval  $[0, (3u+11)/18]$  and decreasing if  $t > (3u+11)/18$ . Since  $\tau(2) = 6u - 14 > \sharp(E)$  and  $\tau(\alpha) \geq 11\alpha > \sharp(E)$ , we get a contradiction.

*Claim 4:* For each  $i \in \{1, \dots, \alpha\}$  we have  $h^1(D_i, \mathcal{I}_{G_i, D_i}(u - 3i + 1, u - 3i + 2)) = 0$ .

*Proof of Claim 4:* We apply Lemma 1 taking  $D_i \in |\mathcal{O}_Q(1, 2)|$  instead of an element of  $|\mathcal{O}_Q(2, 1)|$ . If  $D_i$  is irreducible, then Claim 4 follows from Claim 3, because  $D_i \cong \mathbb{P}^1$  and  $\deg(\mathcal{O}_{D_i}(u - 3i + 1, u - 3i + 2)) = 2(u - 3i + 1) + (u - 3i + 2)$ . If  $D_i$  is a union of 3 lines, then  $b_i \leq 3$ ; in this case we just use that  $u - 3i + 1 > 0$  and  $u - 3i + 2 > 0$ . Now assume  $D_i = T \cup D$  with  $T$  a smooth conic and  $D$  a line. It is sufficient to have  $\sharp(T \cap G_i) \leq 2u - 6i + 4$  (Lemma 1 for curves of type  $(1, 2)$ ). Assume  $\sharp(T \cap G_i) \geq 2u - 6i + 5$ . Since  $T \cup I \in |\mathcal{O}_Q(2, 1)|$  for all  $I \in |\mathcal{O}_Q(1, 0)|$  and  $\sharp(T \cap E_{i-1}) \geq 2u - 6i + 5$ , we get  $a_i \geq 2u - 6i + 6$ . Hence  $\sharp(E) \geq i(4u - 12i + 11)$ . For any  $t \in \mathbb{R}$  set  $\eta(t) := t(4u - 12t + 11)$ . The function  $\eta(t)$  is increasing in the interval  $0 \leq t \leq (4u + 11)/24$  and decreasing if  $t > (4u + 11)/24$ . Since  $\eta(1) = 4u - 1 > 10\alpha$  and  $\eta(\alpha) = \alpha(4u - 12\alpha + 11) \geq 11\alpha$ , we get a contradiction.

By Claims 2 and 4 and the exact sequences (2) and (3) we get  $h^1(\mathcal{I}_E(u, u)) \leq h^1(\mathcal{I}_{E_\alpha}(\beta, \beta))$ . Since  $E_\alpha = \emptyset$ , we have  $h^1(\mathcal{I}_{E_\alpha}(\beta, \beta)) = 0$ .

(ii) Now assume  $v > u$ . Write  $E = F \sqcup F'$  with  $\sharp(F') = \min\{\sharp(E), v - u\}$ . Since  $\sharp(F') \leq v - u$  and no two points of  $E$  are contained in an element of  $|\mathcal{O}_Q(0, 1)|$ , there is a union  $T \subset Q$  of  $v - u$  disjoint elements of  $|\mathcal{O}_Q(0, 1)|$  such that  $F' \subset T$  and  $T \cap F = \emptyset$ . We have an exact sequence

$$(4) \quad 0 \rightarrow \mathcal{I}_F(u, u) \rightarrow \mathcal{I}_E(u, v) \rightarrow \mathcal{I}_{F', T}(u, v) \rightarrow 0$$

Since  $T$  is a disjoint union of  $v - u$  lines, each of them containing at most one point of  $F'$ , we have  $h^1(T, \mathcal{I}_{F', T}(u, v)) = 0$ . Step (i) gives  $h^1(Q, \mathcal{I}_F(u, u)) = 0$ . Hence (4) gives  $h^1(Q, \mathcal{I}_E(u, v)) = 0$ .  $\square$

**Lemma 3.** Fix integers  $x, \alpha, \beta, z$  such that  $\alpha \geq 3$ ,  $\beta \geq 2$ ,  $0 \leq x \leq (\beta + 1)^2$  and  $0 \leq z \leq 10\alpha$ . Set  $u := 3\alpha + \beta$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . Fix  $B \subset Q \setminus S$  such that  $\sharp(B) = z$ , no line of  $Q$  contains 2 points of  $S \cup B$ ,  $\sharp(B \cap T_1) \leq 2u - 2$  for every  $T_1 \in |\mathcal{O}_Q(1, 1)|$ ,  $\sharp(B \cap T_2) \leq 3u - 4$  for every  $T_2 \in |\mathcal{O}_Q(2, 1)|$  and  $\sharp(B \cap T_3) \leq 3u - 9$  for every  $T_3 \in |\mathcal{O}_Q(1, 2)|$ . Then  $h^1(\mathcal{I}_{S \cup B}(u, u)) = 0$ .

*Proof.* Set  $E_0 := S \cup B$  and  $B_0 := B$ . Since  $S$  is general, we have  $\sharp(S \cap T_1) \leq 3$  for every  $T_1 \in |\mathcal{O}_Q(1, 1)|$ ,  $\sharp(S \cap T_2) \leq 5$  for every  $T_2 \in |\mathcal{O}_Q(2, 1)|$  and  $\sharp(S \cap T_3) \leq 5$  for every  $T_3 \in |\mathcal{O}_Q(1, 2)|$  (Remark 1). Hence  $\sharp(E_0 \cap T_1) \leq 2u + 1$  for every  $T_1 \in |\mathcal{O}_Q(1, 1)|$ ,  $\sharp(E_0 \cap T_2) \leq 3u + 1$  for every element of  $|\mathcal{O}_Q(2, 1)|$  and  $\sharp(E_0 \cap T_3) \leq 3u - 4$  for every element of  $|\mathcal{O}_Q(1, 2)|$ . Take any  $A_1 \in |\mathcal{O}_Q(2, 1)|$  such that  $a_1 := \sharp(B_0 \cap A_1)$  is maximal. Set  $F_1 := A_1 \cap E_0$ ,  $B'_1 := A_1 \cap B_0$ ,  $E_{1,0} := E_0 \setminus F_1$  and  $B_{1,0} := B_0 \setminus B'_1$ . Let  $D_1 \in |\mathcal{O}_Q(1, 2)|$  be a curve such that  $b_1 := \sharp(B_{1,0} \cap D_1)$  is maximal. Set  $G_1 := E_{1,0} \cap D_1$ ,  $B''_1 := B_{1,0} \cap D_1$ ,  $B_1 := B_{1,0} \setminus B''_1$  and  $E_1 := E_{1,0} \setminus G_1$ . For each  $i \in \{2, \dots, \alpha\}$  we define recursively the integers  $a_i$ ,  $b_i$ , the curves  $A_i \in |\mathcal{O}_Q(2, 1)|$ ,  $D_i \in |\mathcal{O}_Q(1, 2)|$  and the sets  $F_i$ ,  $E_{i,0}$ ,  $G_i$ ,  $B'_i$ ,  $B_{i,0}$ ,  $B_i$ ,  $B''_i$ ,  $E_i$  in the following way. Take  $A_i \in |\mathcal{O}_Q(2, 1)|$  such that  $a_i := \sharp(A_i \cap B_{i-1})$  is maximal. Set  $F_i := E_{i-1} \cap A_i$ ,

$B'_i := A_i \cap B_{i-1}$ ,  $B_{i,0} := B_{i-1} \setminus B'_i$  and  $E_{i,0} := E_{i-1} \setminus F_i$ . Take  $D_i \in |\mathcal{O}_Q(1, 2)|$  such that  $b_i := \#(D_i \cap B_{i,0})$  is maximal and set  $G_i := E_{i,0} \cap D_i$ ,  $E_i := E_{i,0} \setminus G_i$ ,  $B''_i := B_{i,0} \cap D_i$  and  $B_i := B_{i,0} \setminus B''_i$ . For each  $i \in \{1, \dots, \alpha\}$  we have the exact sequences (2) and (3). Notice that the sequences  $\{a_i\}_{1 \leq i \leq \alpha}$  and  $\{b_i\}_{1 \leq i \leq \alpha}$  are non-increasing. If  $a_i \leq 4$ , then  $E_{i,0} = \emptyset$ , because  $h^0(Q, \mathcal{O}_Q(2, 1)) = 6$ . If  $b_i \leq 4$ , then  $E_i = \emptyset$ , because  $h^0(Q, \mathcal{O}_Q(1, 2)) = 6$ .

*Claim 1:* For every  $i \in \{1, \dots, \alpha\}$  we have  $a_i \leq 3u - 9i + 5$  and  $\#(A_i \cap E_{i-1}) \leq 3u - 9i + 10$ .

*Proof of Claim 1:* Since  $\#(A_i \cap S) \leq 5$ , it is sufficient to prove the inequality  $a_i \leq 3u - 9i + 5$ . Assume  $a_i \geq 3u - 9i + 6$ . Since at most  $3u - 4$  of the points of  $B$  are contained in a curve of type  $(2, 1)$ , we have  $i \geq 2$ . Since the sequences  $\{a_j\}$  and  $\{b_j\}$  are non-increasing and  $b_j \geq 5$  if  $a_{j+1} > 0$ , we get  $\#(B) \geq i(3u - 9i + 6) + 5(i - 1) = i(3u + 11 - 9i) - 5$ . If  $i = 2$ , then we get  $\#(B) \geq 6u - 19 \geq 18\alpha + 6\beta - 19$ , contradicting the assumptions  $\#(B) \leq 10\alpha$ ,  $\alpha \geq 3$  and  $\beta > 0$ . For any  $t \in \mathbb{R}$  set  $\phi(t) = t(3u + 11 - 9t) - 5$ . The function  $\phi$  is increasing in the interval  $[0, (3u + 11)/18]$  and decreasing if  $t \geq (3u + 11)/18$ . Since  $\#(B) < \phi(2)$  and  $\phi(\alpha) = \alpha(3u + 11 - 9\alpha) - 5 = \alpha(3\beta + 11) - 5 > 10\alpha \geq \#(B)$ , we get a contradiction.

*Claim 2:* For each  $i \in \{1, \dots, \alpha\}$  we have  $h^1(A_i, \mathcal{I}_{F_i, A_i}(u - 3i + 3, u - 3i + 3)) = 0$ .

*Proof of Claim 2:* By Claim 1 we have  $\#(F_i) \leq 3u - 9i + 10$ . If  $F_i$  is irreducible, then Claim 2 is true (e.g. by Lemma 1). If  $A_i$  is the union of 3 lines, then  $\#(F_i) \leq 3$  and Claim 2 is true (Lemma 1). Now assume  $A_i = T \cup D$  with  $T$  a smooth conic and  $D$  of type  $(1, 0)$ . By Lemma 1 Claim 2 is true if  $\#(E_{i-1} \cap T) \leq 2u - 6i + 7$ . Assume  $\#(E_{i-1} \cap T) \geq 2u - 6i + 8$ . Since  $\#(S \cap T) \leq 3$ , we get  $\#(B_{i-1} \cap T) \geq 2u - 6i + 5$ . Since  $\#(T_1 \cap B) \leq 2u - 2$  for every  $T_1 \in |\mathcal{O}_Q(1, 1)|$ , we have  $i \geq 2$ . We have  $a_i \geq \#(B_{i-1} \cap T) \geq 2u - 6i + 5$ . Since  $b_j \geq 5$  if  $B_{j+1} \neq \emptyset$ , we get  $\#(B) \geq i(2u - 6i + 5) + 5(i - 1) = i(2u + 10 - 6i) - 5$ . If  $i = 2$ , then  $\#(B) \geq 4u - 9$ , a contradiction. For every  $t \in \mathbb{R}$  set  $\psi(t) := t(2u + 10 - 6t) - 5$ . The function  $\psi$  is increasing in the interval  $[0, (u + 5)/6]$  and decreasing for  $t > (u + 5)/6$ . Since  $\#(B) < \psi(2)$  and  $\psi(\alpha) = \alpha(2u + 10 - 6\alpha) = \alpha(10 + 2\beta) > 10\alpha \geq \#(B)$ , we get a contradiction.

*Claim 3:* For each  $i \in \{1, \dots, \alpha\}$  we have  $b_i \leq 3u - 9i$  and  $\#(G_i) \leq 3u - 9i + 5$ .

*Proof of Claim 3:* Since  $\#(S \cap D_i) \leq 5$ , it is sufficient to prove  $b_i \leq 3u - 9i$ . Assume  $b_i \geq 3u - 9i + 1$ . Since  $G_i \subseteq D_i$ , our assumptions on  $B$  gives  $i \geq 2$ . Since  $b_i > 0$ , we have  $a_j \geq 5$  for all  $j \leq i$ . Hence  $\#(B) \geq 5i + i(3u - 9i + 1) = i(3u + 6 - 9i)$ . Set  $\tau(t) = t(3u + 6 - 9t)$ . The function  $\tau(t)$  is increasing in the interval  $[0, (3u + 6)/18]$  and decreasing if  $t > (3u + 6)/18$ . Since  $\tau(2) = 6u - 24 \geq 18\alpha + 3\beta - 24 > 10\alpha \geq \#(B)$  and  $\tau(\alpha) = \alpha(3u + 6 - 9\alpha) \geq \alpha(6 + 3\beta) \geq 12\alpha$ , we get a contradiction.

*Claim 4:* For each  $i \in \{1, \dots, \alpha\}$  we have  $h^1(D_i, \mathcal{I}_{G_i, D_i}(u - 3i + 1, u - 3i + 2)) = 0$ .

*Proof of Claim 4:* We apply Lemma 1 taking  $D_i \in |\mathcal{O}_Q(1, 2)|$  instead of an element of  $|\mathcal{O}_Q(2, 1)|$ . If  $D_i$  is irreducible, then Claim 4 follows from Claim 3, because  $D_i \cong \mathbb{P}^1$  and  $\deg(\mathcal{O}_{D_i}(u - 3i + 1, u - 3i + 2)) = 2(u - 3i + 1) + (u - 3i + 2)$ . If  $D_i$  is a union of 3 lines, then  $b_i \leq 3$ ; in this case we just use that  $u - 3i + 1 > 0$  and  $u - 3i + 2 > 0$ . Now assume  $D_i = T \cup D$  with  $T$  a smooth conic and  $D$  a line. It is sufficient to have  $\#(G_i \cap T) \leq 2u - 6i + 4$  (Lemma 1 for curves of type  $(1, 2)$ ). Assume  $\#(T \cap G_i) \geq 2u - 6i + 5$ . Since  $S$  is general and  $h^0(\mathcal{O}_Q(1, 1)) = 4$ , we have

$b_i = \sharp(T \cap B_i'') \geq 2u - 6i + 2$ . Since  $T \cup I \in |\mathcal{O}_Q(2, 1)|$  for each  $I \in |\mathcal{O}_Q(1, 0)|$  and  $\sharp(T \cap B_{i-1}) \geq 2u - 6i + 2$ , we have  $a_i \geq 2u - 6i + 3$ . Hence  $a_j \geq 2u - 6i + 3$  for all  $j \leq i$ . Hence  $\sharp(B) \geq i(4u - 12i + 5)$ . Set  $\eta_1(t) := t(4u + 5 - 12t)$ . The function  $\eta_1(t)$  is increasing if  $0 \leq t \leq (4u + 5)/24$  and decreasing if  $t > (4u + 5)/24$ . We have  $\eta_1(1) = 4u - 7 = 12\alpha + 4\beta - 7 > 10\alpha$ . We have  $\eta_1(\alpha) = \alpha(4\beta + 5) \geq 13\alpha > 10\alpha$ . Since  $\sharp(B) \leq 10\alpha$ , we get a contradiction.

By Claims 2 and 4 and the exact sequences (2) and (3) we get  $h^1(\mathcal{I}_E(u, u)) \leq h^1(\mathcal{I}_{E_\alpha}(\beta, \beta))$ . We have  $E_\alpha \subseteq S$ . Since  $\sharp(S) = x \leq (\beta + 1)^2$  and  $S$  is general, we have  $h^1(\mathcal{I}_S(\beta, \beta)) = 0$ . Hence  $h^1(\mathcal{I}_{E_\alpha}(\beta, \beta)) = 0$ .  $\square$

### 3. $d_4(C)$ FOR THE NORMALIZATION $C$ OF A NODAL $Y \subset Q$

For any finite set  $S \subset Q$  let  $2S$  denote the first infinitesimal neighborhood of  $S$  in  $Q$ , i.e. the closed subscheme of  $Q$  with  $(\mathcal{I}_S)^2$  as its ideal sheaf. The scheme  $2S$  is zero-dimensional,  $(2S)_{\text{red}} = S$  and  $\deg(2S) = 3 \cdot \sharp(S)$ .

**Lemma 4.** *Fix integers  $a, b, x$  such that  $b \geq a \geq 4$  and  $0 \leq 3x \leq ab$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . We have  $h^0(Q, \mathcal{I}_{2S}(a, b)) = (a + 1)(b + 1) - 3x$ . Fix a general  $Y \in |\mathcal{I}_{2S}(a, b)|$ . Then  $Y$  is integral, nodal and  $\text{Sing}(Y) = S$ .*

*Proof.* We have  $h^1(Q, \mathcal{I}_{2S}(a - 1, b - 1)) = 0$  ([2], Theorem 1.1). Hence  $h^1(Q, \mathcal{I}_{2S}(a, b)) = 0$ , i.e.  $h^0(Q, \mathcal{I}_{2S}(a, b)) = (a + 1)(b + 1) - 3x$ . Since  $h^1(Q, \mathcal{I}_{2S}(a - 1, b - 1)) = 0$  and the line bundle  $\mathcal{O}_Q(1, 1)$  is very ample, Castelnuovo-Mumford's lemma implies that the sheaf  $\mathcal{I}_{2S}(a, b)$  is spanned. Hence  $|\mathcal{I}_{2S}(a, b)|$  has no base points outside  $S$ . Bertini's theorem implies  $S = \text{Sing}(Y)$ . Fix  $P \in S$  and set  $S' := S \setminus \{P\}$ . Take a general  $D \in |\mathcal{I}_{2S'}(a - 1, b - 1)|$ . Since  $h^1(Q, \mathcal{I}_{2S}(a - 1, b - 1)) = 0$ , we have  $h^1(Q, \mathcal{I}_{2S' \cup \{P\}}(a - 1, b - 1)) = 0$ . Hence  $h^0(\mathcal{I}_{2S' \cup \{P\}}(a - 1, b - 1)) = h^0(\mathcal{I}_{2S'}(a - 1, b - 1)) - 1$ . Since  $D$  is general, we get  $P \notin D$ . Let  $D' \cup D'' \subset Q$  be the reducible conic with  $P$  as its singular locus. Since  $P \notin D$ ,  $D \cup D' \cup D''$  is an element of  $|\mathcal{I}_{2S}(a, b)|$  with an ordinary node at  $P$ . Since  $Y$  is general, it has an ordinary node at  $P$ . Since this is true for all  $P \in S$ ,  $Y$  is nodal. For every irreducible component  $T$  of  $Y$  we have  $\omega_Q \cdot T = \mathcal{O}_Q(-2, -2) \cdot T < 0$ . Since  $b \geq a \geq 4$  and  $3x \leq ab$ , we have  $p_a(Y) = ab - a - b + 1 \geq x$ . Since  $Y$  is nodal, no component of  $Y$  appears with multiplicity  $\geq 2$ . Since  $S$  is general and  $S = \text{Sing}(Y)$ , the curve  $Y$  is irreducible ([1], Proposition 4.1).  $\square$

**Lemma 5.** *Fix integers  $a, b, x$  such that  $b \geq a \geq 4$  and  $0 \leq 3x \leq (a - 1)(b - 1)$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . Fix zero-dimensional schemes  $Z, Z' \subset Q$  such that  $\deg(Z) = \deg(Z') = 2$ ,  $Z_{\text{red}}$  and  $(Z')_{\text{red}}$  are distinct points,  $Z$  is contained in a line  $D_1 \in |\mathcal{O}_Q(1, 0)|$ ,  $Z'$  is contained in a line  $D_2 \in |\mathcal{O}_Q(0, 1)|$  and  $Z \cap D_2 = Z' \cap D_1 = S \cap (D_1 \cup D_2) = \emptyset$ . Take a general  $Y \in |\mathcal{I}_{Z \cup Z' \cup 2S}(a, b)|$ . Then  $h^0(Q, \mathcal{I}_{Z \cup Z' \cup 2S}(a, b)) = (a + 1)(b + 1) - 3x - 4$ ,  $Y$  is nodal, integral,  $\text{Sing}(Y) = S$ ,  $\sharp((Y \cap D_1)_{\text{red}}) = b - 1$  and  $\sharp((Y \cap D_2)_{\text{red}}) = a - 1$ .*

*Proof.* We have  $h^1(Q, \mathcal{I}_{2S}(a - 2, b - 2)) = 0$  ([2], Theorem 1.1). We immediately get  $h^1(Q, \mathcal{I}_{Z \cup Z' \cup 2S}(a, b)) = 0$ , i.e.  $h^0(Q, \mathcal{I}_{Z \cup Z' \cup 2S}(a, b)) = (a + 1)(b + 1) - 3x - 4$ . We also see that  $h^1(Q, \mathcal{I}_{Z \cup Z' \cup 2S}(a - 1, b - 1)) = 0$ . Hence  $\mathcal{I}_{Z \cup Z' \cup 2S}(a, b)$  is spanned by Castelnuovo-Mumford's lemma. Hence  $Y$  is smooth outside  $S \cup \{Z_{\text{red}}, (Z')_{\text{red}}\}$ . Lemma 4 applied to the integers  $a - 1$  and  $b - 1$  gives the existence of an integral and nodal curve  $T \in |\mathcal{I}_{2S}(a - 1, b - 1)|$  such that  $S = \text{Sing}(T)$ . Since  $\mathcal{I}_{2S}(a - 1, b - 1)$  is spanned, we may find  $T$  as above and with  $Z_{\text{red}} \notin T$  and  $(Z')_{\text{red}} \notin T$ . Since

$T \cup D_1 \cup D_2 \in |\mathcal{I}_{Z \cup Z' \cup 2S}(a, b)|$  and  $S \cap (D_1 \cup D_2) = \emptyset$ , we get that  $Y$  is smooth at  $(Z)_{red}$  and at  $(Z')_{red}$  and nodal at each point of  $S$ . Since  $T$  is irreducible and  $Y$  is general, either  $Y$  is irreducible or  $Y = T_1 \cup A_1$  with  $T_1 \in |\mathcal{O}_Q(a, b-1)|$  and  $A_1 \in |\mathcal{O}_Q(0, 1)|$  or  $Y = T_2 \cup A_2$  with  $T_2 \in |\mathcal{O}_Q(a-1, b)|$  and  $A_2 \in |\mathcal{O}_Q(1, 0)|$  or  $Y = T_3 \cup A_3$  with  $T_3 \in |\mathcal{O}_Q(a-1, b-1)|$  and  $A_3 \in |\mathcal{O}_Q(1, 1)|$ . The last three cases are impossible, because  $Y$  is nodal,  $\text{Sing}(Y) = S$ , no  $a-1$  of the points of  $S$  are contained in a line and no conic contains  $a+b-2$  points of  $S$ .

Since  $Z \subseteq D_1 \cap Y$ , we have  $\sharp(Y \cap D_1) \leq b-1$ . Since  $\mathcal{I}_{2S \cup Z \cup Z'}(a, b)$  is spanned and  $Y$  is general,  $Y$  does not contain the degree 3 divisor of  $D_1$  with  $Z_{red}$  as its support. Hence  $Z_{red}$  appears with multiplicity two in the scheme  $Y \cap D_1$ . We need to prove that the other points of  $(Y \cap D_1)_{red}$  appear with multiplicity one in the scheme  $Y \cap D_1$ . Fix  $P \in D_1 \setminus Z_{red}$  and let  $W \subset D_1$  be the divisor of degree two with  $P$  as its support. Since  $h^1(\mathcal{I}_{2S}(a-1, b-1)) = 0$  and  $b \geq a \geq 3$ , we have  $h^1(\mathcal{I}_{2S \cup Z \cup Z' \cup W}(a, b)) = 0$ . Hence  $|\mathcal{I}_{2S \cup Z \cup Z' \cup W}(a, b)|$  has codimension two in  $|\mathcal{I}_{2S \cup Z \cup Z'}(a, b)|$ . Since  $\dim(D_1) = 1$  and  $Y$  is general,  $Y$  contains no such scheme  $W$ . Hence  $\sharp((Y \cap D_1)_{red}) = b-1$ . In the same way we prove that  $\sharp((Y \cap D_2)_{red}) = a-1$ .  $\square$

**Lemma 6.** *Fix integers  $a, m, x$  such that  $a \geq 4$ ,  $0 \leq m < a$  and  $0 \leq 3x \leq (a-1)(a+m-1)$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . Let  $C$  be the normalization of a general  $Y \in |\mathcal{I}_{2S}(a, a+m)|$ . Let  $u_1 : C \rightarrow \mathbb{P}^1$  (resp.  $u_2 : C \rightarrow \mathbb{P}^1$ ) be the  $g_a^1$  (resp.  $g_{a+m}^1$ ) induced by the projection of  $Q$  onto its second (resp. first) factor. Then neither  $u_1$  nor  $u_2$  is composed with an involution, i.e. there are no triple  $(C_i, v_i, w_i)$  with  $C_i$  a smooth curve,  $w_i : C \rightarrow C_i$ ,  $v_i : C_i \rightarrow \mathbb{P}^1$ ,  $u_i = v_i \circ w_i$ ,  $\deg(v_i) > 1$  and  $\deg(w_i) > 1$ .*

*Proof.* It is sufficient to find  $O \in \mathbb{P}^1$  and  $O' \in \mathbb{P}^1$  such that  $u_2^{-1}(O)$  is formed by  $a+m-1$  points, one of them being an ordinary ramification point of  $u_2$ , and  $u_1^{-1}(O')$  is formed by  $a-1$  points, one of them being an ordinary ramification point of  $u_1$ . Fix zero-dimensional schemes  $Z, Z' \subset Q$  such that  $\deg(Z) = \deg(Z') = 2$ ,  $Z_{red}$  and  $(Z')_{red}$  are distinct points,  $Z$  is contained in a line  $D \in |\mathcal{O}_Q(1, 0)|$ ,  $Z'$  is contained in a line  $D' \in |\mathcal{O}_Q(0, 1)|$  and  $S \cap (D \cup D') = \emptyset$ . Take a general  $M \in |\mathcal{I}_{Z \cup Z' \cup 2S}(a, a+m)|$ . Lemma 5 gives that  $M$  is a nodal and irreducible curve,  $\text{Sing}(M) = S$ ,  $\sharp((D \cap M)_{red}) = a+m-1$  and  $\sharp((D' \cap M)_{red}) = a-1$ . Let  $u' : C' \rightarrow M$  be the normalization of  $M$ . Call  $u'_1 : C' \rightarrow \mathbb{P}^1$  and  $u'_2 : C' \rightarrow \mathbb{P}^1$  the pencils induced by the projections of  $Q$ . The scheme  $M \cap D$  is the disjoint union of  $Z$  and  $a+m-2$  distinct points and the scheme  $M \cap D'$  is the disjoint union of  $Z'$  and  $a-2$  distinct points. Since  $(Z')_{red} \cap S = \emptyset$  and  $Z_{red} \cap S = \emptyset$ , there are unique points  $O_1, O'_1 \in C$  such that  $u'(O_1) = Z_{red}$  and  $u'(O'_1) = (Z')_{red}$ . Set  $O' := u_2(O'_1)$  and  $O := u_1(O_1)$ . The  $a-2$  (resp.  $a+m-2$ ) points of  $u_1^{-1}(O') \setminus \{O'_1\}$  (resp.  $u_2^{-1}(O) \setminus \{O_1\}$ ) appear with multiplicity one in the fiber  $u_1^{-1}(O')$ , because  $u$  is a local isomorphism at each of these points and  $D'$  (resp.  $D$ ) is transversal to  $M$  outside  $(Z')_{red}$  (resp.  $Z_{red}$ ). Hence  $O_1$  (resp.  $O'_1$ ) is an ordinary ramification point of  $u'_2$  (resp.  $u'_1$ ). Since  $Y$  is a general equitrivial deformation of  $M$  inside  $|\mathcal{I}_{2S}(a, a+m)|$ , each  $u_i$  has a fiber with a unique ramification point and this ramification point is an ordinary one.  $\square$

**Proposition 1.** *Fix integers  $a, m, x$  such that  $a \geq 18$ ,  $x \geq 0$ ,  $0 \leq m < a$  and*

$$(5) \quad x \leq a/3 + m$$

Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . A general  $Y \in |\mathcal{I}_{2S}(a, a+m)|$  is an integral nodal curve with  $S$  as its singular locus. Let  $u : C \rightarrow Y$  denote the normalization map. We have  $g(C) = a^2 + am - 2a - m + 1 - x$  and  $d_4(C) \geq 3a - 14$ . If  $a \geq 4m + 43$ , then  $d_4(C)/4 > d_3(C)/3$ .

*Proof.* Lemma 5 gives that  $Y$  is integral and nodal and that  $S = \text{Sing}(Y)$ . Hence  $C$  has genus  $a^2 + am - 2a - m + 1 - x$ .

Since the line bundle  $u^*(\mathcal{O}_Y(1, 1))$  has degree  $2a + m$ , we have  $d_3(C) \leq 2a + m$ . Notice that  $4(2a + m) < 3(3a - 14)$  if  $a \geq 4m + 43$ . Set  $z := d_4(C)$  and assume  $z \leq 3a - 15$ .

Fix  $L \in \text{Pic}^z(C)$  evincing  $d_4(C)$ . The line bundle  $L$  is spanned ([3], Lemma 3.1 (b)). Fix a general  $A \in |L|$ . Set  $B := u(A)$ . Since  $L$  has no base points and  $A$  is general,  $S \cap B = \emptyset$ . Since  $Q$  has only finitely many lines intersecting  $S$  and  $A$  is general, we may assume  $B$  disjoint from these finitely many lines. Hence no line of  $Q$  contains a point of  $S$  and at least one point of  $B$ .

*Claim 1:*  $h^1(\mathcal{I}_{S \cup B}(a - 2, a + m - 2)) > 0$ .

*Proof of Claim 1:* Since  $L$  has no base points, we have  $h^0(C, \mathcal{O}_C(A \setminus \{O\})) = h^0(C, \mathcal{O}_C(A)) - 1$  for every  $O \in A$ . Hence  $h^0(C, \omega_C(-A + \{O\})) = h^0(C, \omega_C(-A))$  for every  $O \in A$  (Riemann-Roch and Serre duality). We have  $\omega_Q \cong \mathcal{O}_Q(-2, -2)$ . Hence the adjunction formula gives  $\omega_Y \cong \mathcal{O}_Y(a - 2, a + m - 2)$ . Since  $h^i(\mathcal{O}_Q(-2, -2)) = 0$ ,  $i = 0, 1$ , the restriction map  $H^0(\mathcal{O}_Q(a - 2, a + m - 2)) \rightarrow H^0(Y, \omega_Y)$  is bijective. Since  $Y$  has only ordinary nodes as singularities, we have  $H^0(C, \omega_C) \cong H^0(\mathcal{I}_S(a - 2, a + m - 2))$ . Hence for any  $O \in A$  we have  $h^0(\mathcal{I}_{S \cup (B \setminus \{u(O)\})}(a - 2, a + m - 2)) = h^0(C, \omega_C(-(A \setminus \{O\}))) = h^0(C, \omega_C(-A)) = h^0(\mathcal{I}_{S \cup B}(a - 2, a + m - 2))$ . Hence  $h^1(\mathcal{I}_{S \cup B}(a - 2, a + m - 2)) > 0$ , concluding the proof of Claim 1.

Let  $v_2 : C \rightarrow \mathbb{P}^1$  (resp.  $v_1 : C \rightarrow \mathbb{P}^1$ ) denote the degree  $a$  (resp. degree  $a + m$ ) morphism obtained composing  $u$  with the pencil associated to  $|\mathcal{O}_Q(0, 1)|$  (resp.  $|\mathcal{O}_Q(1, 0)|$ ). Lemma 6 shows that none of these two pencils factors non-trivially.

*Claim 2:* For a general  $B$  no line of  $Q$  contains two or more points of  $B$ .

*Proof of Claim 2:* Assume for instance that for a general  $B$  there is a line  $D_B \in |\mathcal{O}_Q(0, 1)|$  such that  $\sharp(D_B \cap B) \geq 2$ . Set  $\Psi := \{(P, Q) \in C \times C : P \neq Q \text{ and } v_2(P) = v_2(Q)\}$ . For any  $D \in |\mathcal{O}_Q(0, 1)|$  the scheme  $D \cap Y$  is zero-dimensional. Since  $\dim(|L|) = 3 + \dim(|\mathcal{O}_Q(0, 1)|)$ , there is a one-dimensional irreducible set  $\Phi \subseteq \Psi$  such that for all  $(P, Q) \in \Phi$  the set  $\{P, Q\}$  is contained in a 3-dimensional family  $F_{\{P, Q\}}$  of elements of  $|L|$ . Fix  $(P, Q) \in \Phi$ . Since  $L$  has no base points, we have  $h^0(L(-P)) = 4$ . Hence the existence of the family  $F_{\{P, Q\}}$  implies  $h^0(L(-P - Q)) = h^0(L(-P))$ . Hence  $Q$  is a base point of  $|L(-P)|$ . The two projections  $C \times C \rightarrow C$  induce dominant maps  $\Phi \rightarrow C$ . Hence  $P$  may be seen as a general point of  $C$ . Since  $Q \neq P$ , the morphism  $\varphi : C \rightarrow \mathbb{P}^4$  associated to  $|L|$  is not birational onto its image, i.e.  $\varphi = u_2 \circ u_1$  with  $\deg(u_1) \geq 2$ ,  $u_1 : C \rightarrow C'$  a morphism of degree  $\geq 2$  with  $C'$  a smooth curve and  $u_2 : C' \rightarrow \varphi(C) \hookrightarrow \mathbb{P}^4$  birational onto its image. We have  $z = \deg(u_1) \cdot \deg(\varphi(C)) \geq 4 \deg(u_1)$ . Since  $u_1(P) = u_1(Q)$  for a general  $(P, Q) \in \Phi$ , a general fiber of  $u_1$  intersects in at least two points a fiber of  $v_2$ . Since  $v_2$  is not composed with a pencil,  $u_1$  factors through  $v_2$ . Hence  $z \geq 4a$ , a contradiction. Hence Claim 2 is true.

Since  $S$  is finite, there are only finitely many lines of  $Q$  containing at least one point of  $S$ . Call  $\Gamma$  their union. Since  $S$  is general, no such a line contains at least two points of  $S$ . Since  $|L|$  has no base points, and  $\Gamma \cap Y$  is finite, for general  $B$  we may assume  $B \cap \Gamma = \emptyset$ . Hence Claim 2 implies that no line of  $Q$  contains at least



two points of  $S \cup B$ . Claim 1 gives  $h^1(Q, \mathcal{I}_{S \cup B}(a-2, a+m-2)) > 0$ . By (5) we have  $x+z \leq 3a-15+a/3+m \leq 10\lfloor a/3 \rfloor + 20/3 + m - 15 \leq 10\lfloor a/3 \rfloor + m - 8$ . To apply Lemma 2 with  $E = S \cup B$ ,  $a = u$  and  $v = a+m$  and get a contradiction it is sufficient to prove that  $\sharp((S \cup B) \cap T_1) \leq 2a-3$  for all  $T_1 \in |\mathcal{O}_Q(1,1)|$ ,  $\sharp((S \cup B) \cap T_2) \leq 3(a-2)+1$  for every  $T_2 \in |\mathcal{O}_Q(2,1)|$  and  $\sharp((S \cup B) \cap T_2) \leq 3(a-2)-4$  for every  $T_2 \in |\mathcal{O}_Q(2,1)|$ . Fix  $T_1 \in |\mathcal{O}_Q(1,1)|$ ,  $T_2 \in |\mathcal{O}_Q(2,1)|$  and  $T_3 \in |\mathcal{O}_Q(1,2)|$ . Set  $y_i := \sharp(S \cap T_i)$  and  $a_i := \sharp(B \cap T_i)$ . Assume either  $y_1 + a_1 \geq 2a-2$  or  $y_2 + a_2 \geq 3a-4$  or  $y_3 + a_3 \geq 3a-9$ . Since  $S$  is general, we have  $y_1 \leq 3$ ,  $y_2 \leq 5$  and  $y_3 \leq 5$ . Hence either  $a_1 \geq 2a-5$  or  $a_2 \geq 3a-9$  or  $a_3 \geq 3a-14$ . Since  $z \geq a_i$  for every  $i$  such that  $T_i$  exists and  $z \leq 3a-15$ , we get the existence of  $T_1 \in |\mathcal{O}_Q(1,1)|$  such that  $\sharp(B \cap T_1) \geq 2a-5$ . Since any line of  $Q$  contains at most one element of  $B$ ,  $T_1$  is irreducible.

Step ( $\diamond$ ) (Proof due to the referee; a simpler form would also prove Claim 2) Let  $\varphi : C \rightarrow \mathbb{P}^4$  be the morphism defined by  $|L|$ . Set  $\Gamma := \varphi(C)$ . Let  $C'$  be the normalization of  $\Gamma$  and let  $f : C \rightarrow C'$  be the covering induced by  $\varphi$ . Let  $g_{z'}^4$ ,  $z' = z/\deg(f)$ , be the linear series on  $C'$  induced by the inclusion  $\Gamma \hookrightarrow \mathbb{P}^4$ . Take  $A' \in g_{z'}^4$  with  $A = f^{-1}(A')$ . Since  $A \in |L|$  is general,  $A'$  is general. The monodromy group of the general hyperplane section of  $\Gamma$  is the full symmetric group. Hence any 4 points of a general hyperplane section of  $\Gamma$  span a 3-dimensional projective space. Hence for any  $E \subset A'$  with  $\sharp(E) \geq 4$ ,  $A'$  is the only element of  $g_{z'}^4$  containing  $E$ . For any  $T \in |\mathcal{O}_Q(1,1)|$  the set  $T \cap Y$  is finite. Since  $\dim(|\mathcal{O}_Q(1,1)|) < 4$ , there is an infinite family  $\mathcal{F} \subset |L|$  such that  $u(D)$  contains  $B \cap T_1$  for all  $D \in \mathcal{F}$ . Fix  $D' \in g_{z'}^4$  such that  $D := f^{-1}(D') \in \mathcal{F}$  and  $D' \neq A'$ . Since  $u(f^{-1}(D' \cap A'))$  contains  $B \cap T_1$ , there is  $P \in A'$  such that  $\sharp(u(f^{-1}(P)) \cap (B \cap T_1)) \geq (2a-5)/3$ . Since  $(2a-5)/3 \geq 3$ ,  $T_1$  is the only element of  $|\mathcal{O}_Q(1,1)|$  containing  $u(f^{-1}(P)) \cap (B \cap T_1)$ . Moving  $A'$  generally in  $g_{z'}^4(-P)$  we get that  $D'$  and  $B \cap T_1$  moves into a subscheme  $B'$  of  $u(f^{-1}(D'))$  contained in some  $T' \in |\mathcal{O}_Q(1,1)|$ . Since  $P \in D'$  and  $\sharp(u(f^{-1}(P)) \cap (B \cap T_1)) \geq 3$ , we have  $T' = T_1$ . Hence  $B \cap T_1$  does not move moving  $A'$  in  $g_{z'}^4(-P)$ . Hence  $\sharp(f^{-1}(P)) \geq 2a-5$ , i.e.  $\deg(f) \geq 2a-5$ . Since  $z' \geq 4$ , we get  $z \geq 4(2a-5)$ , a contradiction.  $\square$

**Proposition 2.** *Fix integers  $x, \alpha, \gamma$  such that  $\alpha \geq 3$ ,  $\gamma \geq 4$ ,  $0 \leq x \leq (\gamma-1)^2$ . Set  $a := 3\alpha + \gamma$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$  and a general  $Y \in |\mathcal{I}_{2S}(a, a)|$ .  $Y$  is integral, nodal and  $\text{Sing}(Y) = S$ . Let  $C$  be the normalization of  $Y$ . Then  $d_4(C) \geq \min\{10\alpha + 1, 3a - 14\}$ .*

*Proof.* Set  $z := d_4(C)$  and assume  $z \leq 3a-15$  and  $z \leq 10\alpha$ . Take the set-up of the proof of Proposition 1 with  $m = 0$ . In particular we get a finite set  $B \subset Q \setminus S$  such that  $\sharp(B) = z$ ,  $h^1(Q, \mathcal{I}_{S \cup B}(a-2, a-2)) > 0$  and no line of  $Q$  contains two points of  $S \cup B$ . To get a contradiction we cannot apply Lemma 2 with  $u = v = a-2$  and  $E := S \cup B$ , because  $x$  may be large. We need to check that we may apply Lemma 3 with  $u = a-2$  and  $\beta = \gamma-2$ , i.e. we need to check that no line of  $Q$  contains two points of  $S \cup B$ ,  $\sharp(B \cap T_1) \leq 2a-6$  for every  $T_1 \in |\mathcal{O}_Q(1,1)|$ ,  $\sharp(B \cap T_2) \leq 3a-10$  for every  $T_2 \in |\mathcal{O}_Q(2,1)|$  and  $\sharp(B \cap T_3) \leq 3a-14$  for every  $T_3 \in |\mathcal{O}_Q(1,2)|$ . Since  $z \leq 3a-15$ , we only need to test the conditions for the lines of  $Q$  and that  $\sharp(B \cap T_1) \leq 2a-6$  for each  $T_1 \in |\mathcal{O}_Q(1,1)|$ . Step ( $\diamond$ ) of the proof of Proposition 1 proves the condition for  $T_1 \in |\mathcal{O}_Q(1,1)|$ . We may also copy the proof of Claim 2 of the proof of Proposition 1, because the assumptions of Lemma 5 are satisfied.  $\square$

**Lemma 7.** *Fix integers  $a, x$  such that  $a \geq 24$  and  $0 \leq x \leq 2a - 4$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$  and a general  $Y \in |\mathcal{I}_{2S}(a, a)|$ .  $Y$  is integral, nodal and  $\text{Sing}(Y) = S$ . Let  $C$  be the normalization of  $Y$ . Then  $2a - 5 \leq d_3(C) \leq 2a$ .*

*Proof.* Lemma 4 gives that  $Y$  is integral, nodal and smooth outside  $S$ . The pull-back of the line bundle  $\mathcal{O}_Q(1, 1)$  gives  $d_3(C) \leq 2a$ . Assume  $z := d_3(C) \leq 2a - 6$  and fix  $L \in \text{Pic}^z(C)$  evincing  $d_3(C)$ . As in Claims 1 and 2 of the proof of Proposition 1 we get a set  $B \subset Q \setminus S$  such that  $\sharp(B) = z$ , no line of  $Q$  contains 2 of the points of  $S \cup B$  and  $h^1(\mathcal{I}_{S \cup B}(a - 2, a - 2)) > 0$ . Since  $z \leq 2a - 6$  and  $a \geq 16$ , we have  $z + 5 \leq 3a - 15$ . Hence  $\sharp(T \cap (S \cup B)) \leq 3a - 15$  for every  $T \in |\mathcal{O}_Q(2, 1)|$  and every  $T \in |\mathcal{O}_Q(1, 2)|$ . Since  $z \leq 2a - 6$ , we have  $\sharp(B \cap T) \leq 2(a - 2) - 2$  for every  $T \in |\mathcal{O}_Q(1, 1)|$ . Set  $\gamma := \max\{2, -1 + \sqrt{x}\}$ ,  $\alpha := \lfloor (a - 2 - \gamma)/3 \rfloor$  and  $\beta := a - 2 - 3\alpha$ . We have  $\beta \geq \gamma \geq 2$  and  $x \leq (\gamma + 1)^2 \leq (\beta + 1)^2$ . To apply Lemma 3 (and hence to get  $h^1(\mathcal{I}_{S \cup B}(a - 2, a - 2)) = 0$ , i.e. a contradiction) it is sufficient to prove that  $z \leq 10\alpha$ . This is true if  $x \leq 8$ , because in this case  $\gamma = 2$ . Hence we may assume  $\gamma = -1 + \sqrt{x}$ . Hence  $\alpha \geq (a - 4 - \sqrt{2a})/3$ . We have  $10(a - 4 - \sqrt{2a})/3 \geq 2a - 6$  if and only if  $4a - 22 \geq 10\sqrt{2a}$ . Hence it is sufficient to assume  $a \geq 24$ .  $\square$

**Remark 2.** Take  $C$  as in Proposition 2. Set  $M := u^*(\mathcal{O}_Y(2, 1)) \in \text{Pic}^{3a}(C)$ . Since  $h^0(Q, \mathcal{O}_Q(2, 1)) = 6$  and  $Y$  is contained in no element of  $|\mathcal{O}_Q(2, 1)|$ , we have  $h^0(M) \geq 6$ . Fix  $P \in Y$  such that  $P \in \text{Sing}(Y)$  if  $x > 0$ . Let  $F \subset C$  be the scheme-theoretic pull-back of the scheme  $P$ . We have  $\deg(F) \geq 1 + \min\{1, x\}$ . Since  $h^0(Q, \mathcal{I}_E(2, 1)) = 4$ , we have  $h^0(M(-F)) \geq 5$ . Hence  $d_4(C) \leq 3a - 1 - \min\{1, x\}$ .

**Corollary 1.** *Fix integers  $a, x$  such that  $a \geq 204$  and  $0 \leq x \leq 2a - 4$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$  and a general  $Y \in |\mathcal{I}_{2S}(a, a)|$ .  $Y$  is integral, nodal and  $\text{Sing}(Y) = S$ . Let  $C$  be the normalization of  $Y$ . Then  $2a - 5 \leq d_3(C) \leq 2a$  and  $3a - 15 \leq d_4(C) \leq 3a - 1 - \min\{1, x\}$ .*

*Proof.* Since  $3x \leq 6a - 12 \leq a^2$ ,  $Y$  is integral, nodal and  $\text{Sing}(Y) = S$  (Lemma 4). Lemma 7 gives  $2a - 5 \leq d_3(C) \leq 2a$ . Remark 2 gives  $d_4(C) \leq 3a - 1 - \min\{x, 1\}$ . Assume  $z := d_4(C) \leq 3a - 16$ . Take  $B$  as in the proofs of Proposition 1 and 2. We have  $h^1(\mathcal{I}_{S \cup B}(a - 2, a - 2)) > 0$ . Set  $\delta := -1 + \lceil \sqrt{2a - 4} \rceil$  and  $\alpha := \lfloor (a - 2 - \delta)/3 \rfloor$ . Notice that  $x \leq 2a - 4 \leq (\delta + 1)^2$  and that  $\alpha \geq (a - 5 - \sqrt{2a})/3$ . Since  $a \geq 204$ , we have  $a - 2 \geq 10\sqrt{2a}$ , i.e.  $10(a - 5 - \sqrt{2a})/3 \geq 3a - 16$ . Hence  $10\alpha \geq 3a - 16$ . By assumption we have  $d_3(C) \leq 3(a - 2) - 9$ . Claim 2 and Step  $(\diamond)$  of the proof of Proposition 1 show that we may apply Lemma 3 with the integers  $u := a - 2$  and  $\beta := a - 2 - 3\alpha$  (notice that  $\beta \geq \delta$ ) and get  $h^1(\mathcal{I}_{S \cup B}(a - 2, a - 2)) = 0$ , a contradiction.  $\square$

*Proof of Theorem 1.* For all integers  $a, x$  set  $g_a := a^2 - 2a + 1$  and  $g_{a,x} = g_a - x$ . If  $a > 0$ , then  $g_a = p_a(Y)$  for any  $Y \in |\mathcal{O}_Q(a, a)|$ . Hence if  $Y$  is a nodal curve of type  $(a, a)$  with exactly  $x$  nodes, then  $g_{a,x}$  is the genus of the normalization of  $Y$ . Now assume  $a \geq 2$ . We have  $g_a - g_{a-1} = a^2 - 2a + 1 - (a-1)^2 + 2(a-1) - 1 = 2a - 3$ . Hence the set  $\{g_{a,x}\}_{0 \leq x \leq 2a-4}$  contains every integer between  $g_{a-1} + 1$  and  $g_a$ . We take the set-up of the proof of Corollary 1. Fix an integer  $g \geq 40805$ . Let  $a$  be the only integer such that  $g_{a-1} < g \leq g_a$ . Since  $g_{203} = 40804$ , we have  $a \geq 204$ . We have  $g = g_a - x$  with  $0 \leq x \leq 2a - 4$ . Apply Corollary 1.  $\square$

Of course, the lower bound  $g \geq 40805$  is not sharp.

*Proof of Theorem 2.* For any  $g < 40805$  we take as  $C_g$  an arbitrary smooth curve of genus  $g$ . Fix an integer  $g \geq 40805$  and call  $a$  the minimal positive integer such that  $g \leq a^2 - 2a + 1$ . Set  $x := a^2 - 2a + 1 - g$ . Since  $g > (a-1)^2 - 2(a-1) + 1$ , we have  $x \leq 2a - 4$ . Fix a general  $S \subset Q$  such that  $\sharp(S) = x$ . Take as  $C_g$  the normalization of a general  $Y \in |\mathcal{I}_{2S}(a, a)|$ . Corollary 1 gives  $2a - 6 \leq d_3(C) \leq 2a$  and  $3a - 15 \leq d_4(C) \leq 3a - 1 - \min\{1, x\}$ . We have  $g_{a-1} < g \leq g_a$ . Hence the limits are as in the statement of Theorem 2.  $\square$

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